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Aharonov–Bohm scattering on two parallel flux lines of the same magnitude by the method of path integration

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Abstract. The problem of Aharonov–Bohm (AB) scattering on two parallel flux lines of the same magnitude is solved by the method of path integration.

1. Introduction

We have already derived the AB scattering cross section by the method of directly solving the Schrödinger equation for the case of two parallel magnetic flux lines of the same magnitude [1, 2]. We also gave a comment on the criticism of this method of calculation [3], showing the correctness of our calculation. In this paper we shall rederive the above result by the path integral method. This calculation shows the powerfulness of the path integral method and the correctness of our previous calculation about this problem. In section 2 we obtain the radial propagator in elliptic cylindrical coordinates by the polygonal paths approach. In section 3 we give the complete propagator for this case. In section 4 we obtain an expression for the propagator in the region far from the magnetic flux lines. In section 5 we give the expression for an incident plane wave in elliptic cylindrical coordinates. In section 6 we derive the expression for the wavefunction in the region far from the magnetic flux lines, this expression is the same as that obtained in [1, 2] and hence we obtain the same expression for the AB scattering cross section as before.

2. The radial propagator in elliptic cylindrical coordinates

The propagator

$$K(\boldsymbol{x}'', t''|\boldsymbol{x}', t') = \int \exp\left\{\frac{\mathrm{i}}{\hbar} S[\boldsymbol{x}(t)]\right\} D[\boldsymbol{x}(t)] \tag{1}$$

represents the quantum mechanical transition amplitude for a particle to be found at position x'' at time t'' given that the particle was at position x' at an earlier time t'. Where $S[\boldsymbol{x}(t)]$ is the classical action and $D[\boldsymbol{x}(t)]$ represents the measure of path integration. In the discrete form in rectangular coordinates

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$$\exp\left[\frac{\mathrm{i}}{\hbar}S\right] = \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N} S(x_j, x_{j-1})\right]$$

$$= \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left\{\frac{m}{2\varepsilon}[(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2] - \varepsilon V(x_j, y_j)\right\}\right]. \tag{2}$$

Firstly we use elliptic cylindrical coordinates (μ, θ)

$$x = a\cosh\mu\cos\theta, \qquad y = a\sinh\mu\sin\theta \tag{3}$$

to evaluate the radial propagator (along the direction of the unit coordinate vector $\boldsymbol{e}_\mu$) by the polygonal paths approach. Substituting (3) into (2), using the formula

$$\exp(u\cos\theta) = \sum_{l=-\infty}^{\infty} \exp(\mathrm{i}l\theta) I_l(u) \tag{4}$$

where $I_l(u)$ is the modified Bessel function, assuming the potential term V to be cylindrically symmetric and taking $\sin\theta_j \approx \sin\theta_{j-1}$, we obtain

$$\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N} S(\mu_j, \mu_{j-1})\right] = \prod_{j=1}^{N}\sum_{l=-\infty}^{\infty} \exp[\mathrm{i}l(\theta_j - \theta_{j-1})]$$

$$\times \exp\left\{\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta_j[1 - \cosh(\mu_j - \mu_{j-1})]\right\} R_l(\mu_j, \mu_{j-1}) \tag{5}$$

where

$$R_l(\mu_j, \mu_{j-1}) = \exp\left[\frac{\mathrm{i}ma^2}{2\varepsilon\hbar}(\cosh^2\mu_j + \cosh^2\mu_{j-1}) - \frac{\mathrm{i}\varepsilon}{\hbar}V(\mu_j)\right]$$

$$\times I_l\left(\frac{ma^2}{\mathrm{i}\varepsilon\hbar}\cosh\mu_j\cosh\mu_{j-1}\right). \tag{6}$$

Noting that the area element

$$\mathrm{d}x_j\,\mathrm{d}y_j = a^2(\cosh^2\mu_j - \cos^2\theta_j)\,\mathrm{d}\mu_j\,\mathrm{d}\theta_j \tag{7}$$

using (5)–(7) to rewrite (1) and interchanging the order of the multiplication symbol $\prod$ and the summation symbol $\sum$, we get

$$K = \lim_{N\to\infty} K_N \tag{8}$$

where

$$K_N = B_N \sum_{l_1,\ldots l_N}\int\int\prod_{j=1}^{N}\left\{\exp[\mathrm{i}l_j(\theta_j - \theta_{j-1})]\exp\left(\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta_j[1 - \cosh(\mu_j - \mu_{j-1})]\right)\right\}$$

$$\times R_{l_j}(\mu_j, \mu_{j-1})\prod_{j=1}^{N-1} a^2(\cosh^2\mu_j - \cos^2\theta_j)\,\mathrm{d}\mu_j\,\mathrm{d}\theta_j \tag{9}$$

and

$$B_N = \left(\frac{-\mathrm{i}\beta}{2\pi}\right)^N \qquad \beta = \frac{m}{\varepsilon\hbar}. \tag{10}$$

Integrating (9) with respect to $\theta_j (j = 1, 2, \ldots, N-1)$, we can rewrite (9) as

$$K_N = B_N \sum_{l_1,\ldots,l_N} \int \exp\left\{\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta_N[1-\cosh(\mu_N-\mu_{N-1})]\right\} \times \prod_{j=1}^{N} R_{l_j}(\mu_j,\mu_{j-1}) \prod_{j=1}^{N-1} I_{\mu_j}\,\mathrm{d}\mu_j \tag{11}$$

where

$$I_{\mu_j} = \mathrm{e}^{\mathrm{i}(l_N\theta_N - l_1\theta_0)} \int \prod_{j=1}^{N-1} \exp[\mathrm{i}\theta_j(l_j-l_{j+1})] \exp\left\{\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta_j[1-\cosh(\mu_j-\mu_{j-1})]\right\} \times a^2(\cosh^2\mu_j - \cos^2\theta_j)\,\mathrm{d}\theta_j. \tag{12}$$

Taking $\theta_N = \theta''$, $\theta_0 = \theta'$ and considering $[1-\cosh(\mu_j-\mu_{j-1})]$ as a small quantity δ, then expanding the exponential terms, we obtain

$$I_{\mu_j} = \mathrm{e}^{\mathrm{i}(l_N\theta'' - l_1\theta')} \int \prod_{j=1}^{N-1} \exp[\mathrm{i}\theta_j(l_j-l_{j+1})] \times \left\{1 + \frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta_j[1+\cosh(\mu_j-\mu_{j-1})] + \mathrm{O}(\delta^2)\right\} \times a^2(\cosh^2\mu_j - \cos^2\theta_j)\,\mathrm{d}\theta_j. \tag{13}$$

In order to do the above integration, we quote the following formulae:

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\theta_j}\,\mathrm{d}\theta_j = 2\pi\delta_{k,0}$$

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\theta_j}\cos^2\theta_j\,\mathrm{d}\theta_j = 2\pi\left(\tfrac{1}{2}\delta_{k,0} + \tfrac{1}{4}\delta_{k,2} + \tfrac{1}{4}\delta_{k,-2}\right)$$

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\theta_j}\sin^2\theta_j\cos^2\theta_j\,\mathrm{d}\theta_j = 2\pi\left(\tfrac{1}{8}\delta_{k,0} - \tfrac{1}{16}\delta_{k,4} - \tfrac{1}{16}\delta_{k,-4}\right)$$

where

$$\delta_{a,b} = \begin{cases} 1 & \text{when } a = b \\ 0 & \text{when } a \neq b. \end{cases} \tag{14}$$

From (14) we can see that the non-zero terms appear only when $k = 0, \pm 2, \pm 4$. Taking

$$l_1 = l \qquad l_j - l_{j+1} = k_j = k \qquad (j = 1, 2, \ldots, N-1) \tag{15}$$

we get

$$\mathrm{e}^{\mathrm{i}(l_N\theta'' - l_1\theta')} = \mathrm{e}^{\mathrm{i}l(\theta''-\theta')} \prod_{j=1}^{N-1} \mathrm{e}^{-\mathrm{i}k_j\theta''}. \tag{16}$$

Substituting (14)–(16) into (13) we get

$$I_{\mu_j} = \prod_{j=1}^{N-1} 2\pi a^2 \left\{\cosh^2\mu_j - \frac{1}{2} - \frac{1}{4}(\mathrm{e}^{\mathrm{i}2\theta''} + \mathrm{e}^{-\mathrm{i}2\theta''}) + \mathrm{O}(\delta^2) + \frac{\mathrm{i}ma^2}{\varepsilon\hbar}[1-\cosh(\mu_j-\mu_{j-1})] \times \left[\cosh^2\mu_j\left(\frac{1}{2} - \frac{1}{4}\mathrm{e}^{\mathrm{i}2\theta''} - \frac{1}{4}\mathrm{e}^{-\mathrm{i}2\theta''}\right) - \frac{1}{8} + \frac{1}{16}(\mathrm{e}^{\mathrm{i}4\theta''} + \mathrm{e}^{-\mathrm{i}4\theta''})\right]\right\}$$

$$= (2\pi)^{N-1}\prod_{j=1}^{N-1}\left\{1+\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta''[1-\cosh(\mu_j-\mu_{j-1})]+\mathrm{O}(\delta^2)\right\}$$
$$\times a^2(\cosh^2\mu_j-\cos^2\theta''). \tag{17}$$

Considering that δ is a small quantity, we can return the terms in the curly bracket of (17) to the exponential form. Substituting the obtained result into (11), considering that $\prod_{j=1}^{N} R_{l_j}(\mu_j,\mu_{j-1})$ can be rewritten as $\prod_{j=1}^{N} R_l(\mu_j,\mu_{j-1})$ after interchanging the order of the summation symbol and multiplication symbol, we can write $\sum_{l_1,\ldots,l_N}$ as $\sum_l$, thus we obtain

$$K_N = (2\pi)^{N-1}B_N\sum_l \mathrm{e}^{\mathrm{i}l(\theta''-\theta')}\int\prod_{j=1}^{N}R_l(\mu_j,\mu_{j-1})$$
$$\times\exp\left\{\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta''[1-\cosh(\mu_j-\mu_{j-1})]\right\}$$
$$\times\prod_{j=1}^{N-1}a^2(\cosh^2\mu_j-\cos^2\theta'')\,\mathrm{d}\mu_j. \tag{18}$$

Substituting (18) into (8) and taking $\mu_N=\mu''$, $\mu_0=\mu'$, we get

$$K=\lim_{N\to\infty}K_N=\sum_{l=-\infty}^{\infty}K_l(\mu'',t'',\mu',t')\exp[\mathrm{i}l(\theta''-\theta')] \tag{19}$$

where K_l is the required radial propagator

$$K_l=\lim_{N\to\infty}(2\pi)^{N-1}B_N\int\prod_{j=1}^{N}R_l(\mu_j,\mu_{j-1})\exp\left\{\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta''[1-\cosh(\mu_j-\mu_{j-1})]\right\}$$
$$\times\prod_{j=1}^{N-1}a^2(\cosh^2\mu_j-\cos^2\theta'')\,\mathrm{d}\mu_j. \tag{20}$$

3. The complete propagator

Imposing the constraint

$$\int_{t'}^{t''}\mathrm{d}t\,\dot\theta=\phi=\theta''-\theta'+2\pi n \qquad (n=0,\pm1,\pm2,\ldots) \tag{21}$$

we obtain the constrained propagator

$$K_\phi=\int\delta\left(\phi-\int_{t'}^{t''}\mathrm{d}t\,\dot\theta\right)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}L\,\mathrm{d}t\right]D[E(t)]$$
$$=\int\frac{1}{2\pi}\int\mathrm{d}\lambda\exp\left\{\mathrm{i}\lambda\left(\phi-\int_{t'}^{t''}\mathrm{d}t\,\dot\theta\right)\right\}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}L\,\mathrm{d}t\right]D[E(t)]$$
$$=\frac{1}{2\pi}\int\mathrm{d}\lambda\exp(\mathrm{i}\lambda\phi)K_\lambda(\mu'',t''|\mu',t') \tag{22}$$

where $E(t)$ denotes the path of the particle in elliptic cylindrical coordinates, and

$$K_\lambda(\mu'',t''|\mu',t')=\int\int\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathrm{d}t\,(L-\lambda\hbar\theta)\right]a^2(\cosh^2\mu-\cos^2\theta)\,\mathrm{d}\mu\,\mathrm{d}\theta. \tag{23}$$

Using (4)–(6) and the approximate formulae

$$\cos(\theta_j - \theta_{j-1}) + \alpha\varepsilon(\theta_j - \theta_{j-1}) - \frac{1}{4}\alpha^2\varepsilon^2(\theta_j - \theta_{j-1})^2 \approx \cos(\theta_j - \theta_{j-1} - \alpha\varepsilon) + \frac{1}{2}\alpha^2\varepsilon^2 \tag{24}$$

$$I_m\left(\frac{u}{\varepsilon}\right) = \sqrt{\frac{\varepsilon}{2\pi u}}\exp\left[\frac{u}{\varepsilon} - \frac{(m^2 - \frac{1}{4})\varepsilon}{2u} + \mathrm{O}(\varepsilon^2)\right] \tag{25}$$

we obtain

$$\begin{aligned}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathrm{d}t\,(L - \lambda\hbar\dot{\theta})\right] &= \prod_{j=1}^{N}\sum_{l=-\infty}^{\infty}\exp[\mathrm{i}l(\theta_j - \theta_{j-1})] \\ &\quad\times\exp\left[\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta_j[1 - \cosh(\mu_j - \mu_{j-1})\right]R_{l+\lambda}(\mu_j, \mu_{j-1}).\end{aligned} \tag{26}$$

Substituting (26) into (23), discretizing and integrating with respect to θ_j, we get

$$K_\lambda(\mu'', t''|\mu', t') = \sum_{l=-\infty}^{\infty}\exp[\mathrm{i}l(\theta'' - \theta')]\tilde{K}_{l+\lambda}(\mu'', t''|\mu', t') \tag{27}$$

where

$$\begin{aligned}\tilde{K}_{l+\lambda} = \lim_{N\to\infty}(2\pi)^{N-1}B_N\int\prod_{j=1}^{N}R_{l+\lambda}(\mu_j, \mu_{j-1})\exp\left\{\frac{\mathrm{i}ma^2}{\varepsilon\hbar}\sin^2\theta''[1 - \cosh(\mu_j - \mu_{j-1})]\right\} \\ \times\prod_{j=1}^{N-1}a^2(\cosh^2\mu_j - \cos^2\theta'')\,\mathrm{d}\mu_j.\end{aligned} \tag{28}$$

Substituting (27) into (22) we obtain

$$\begin{aligned}K_\phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\sum_{l=-\infty}^{\infty}\exp[\mathrm{i}l(\theta'' - \theta') + \mathrm{i}\lambda\phi]\tilde{K}_{l+\lambda}(\mu'', t''|\mu', t') \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\sum_{l=-\infty}^{\infty}\exp[\mathrm{i}l(\theta'' - \theta' - \phi) + \mathrm{i}\lambda\phi]\tilde{K}_\lambda(\mu'', t''|\mu', t').\end{aligned} \tag{29}$$

Using the identity

$$\sum_{k=-\infty}^{\infty}\exp(\mathrm{i}k\theta) = 2\pi\sum_{n=-\infty}^{\infty}\delta(\theta + 2\pi n) \tag{30}$$

to rewrite (29), we get

$$K_\phi = \sum_{n=-\infty}^{\infty}\delta(\theta'' - \theta' - \phi + 2\pi n)\int_{-\infty}^{\infty}\exp(\mathrm{i}\lambda\phi)\tilde{K}_\lambda(\mu'', t''|\mu', t')\,\mathrm{d}\lambda. \tag{31}$$

The delta function is effective in picking out the homotopic paths with winding number n. The complete propagator then takes the form

$$K(\mu'', t''|\mu', t') = \int_{-\infty}^{\infty}K_\phi\,\mathrm{d}\phi = \sum_{n=-\infty}^{\infty}K_n(\mu'', t''|\mu', t') \tag{32}$$

where

$$K_n(\mu'', t''|\mu', t') = \int_{-\infty}^{\infty}\exp[\mathrm{i}\lambda(\theta'' - \theta' + 2\pi n)]\tilde{K}_\lambda(\mu'', t''|\mu', t')\,\mathrm{d}\lambda \tag{33}$$

is the homotopic propagator. Noting that

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{e}{c}\boldsymbol{A}\cdot\boldsymbol{r} = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - 2\alpha\hbar\dot{\theta} \qquad \alpha = \frac{-e\Phi}{2\pi\hbar c} \tag{34}$$

(Φ, magnetic flux) finally we obtain the expression of the complete propagator

$$K(\mu'', \theta'', t'', \mu', \theta', t') = \exp[-\mathrm{i}2\alpha(\theta'' - \theta')] \times \sum_{n=-\infty}^{\infty} \exp[-\mathrm{i}4\alpha\pi n] \int_{-\infty}^{\infty} \exp[\mathrm{i}\lambda(\theta'' - \theta' + 2\pi n)]\tilde{K}_\lambda \,\mathrm{d}\lambda \tag{35}$$

where

$$\tilde{K}_\lambda = \lim_{N\to\infty} (2\pi)^{N-1} B_N \int \prod_{j=1}^{N} \exp\left[\frac{\mathrm{i}ma^2}{2\varepsilon\hbar}\{\cosh^2\mu_j + \cosh^2\mu_{j-1} + 2\sin^2\theta''[1 - \cosh(\mu_j - \mu_{j-1})]\}\right] \times I_\lambda\left(\frac{ma^2\cosh\mu_j\cosh\mu_{j-1}}{\mathrm{i}\varepsilon\hbar}\right) \prod_{j=1}^{N-1} a^2(\cosh^2\mu_j - \cos^2\theta'')\,\mathrm{d}\mu_j. \tag{36}$$

4. Expression of the propagator $K(\mu, \theta, T; \mu_0, \theta_0, 0)$ at large distances from two fluxes

In the region far from two magnetic flux lines we have

$$\mu_j \gg 1 \qquad \sinh\mu_j \approx \cosh\mu_j \gg 1 \tag{37}$$

hence we can introduce two small quantities q_{1j} and q_{2j},

$$q_{1j} = \mathrm{i}\beta a^2[\cosh(\mu_j - \mu_{j-1}) - 1] \qquad 1 - q_{2j}\cos^2\theta = \frac{\cosh^2\mu_j - \cos^2\theta}{\cosh\mu_j\sinh\mu_j} \tag{38}$$

and express part of (36) in terms of these quantities as follows:

$$\prod_{j=1}^{N} \exp\{\mathrm{i}\beta a^2\sin^2\theta[1 - \cosh(\mu_j - \mu_{j-1})]\} \prod_{j=1}^{N-1} \frac{\cosh^2\mu_j - \cos^2\theta}{\cosh\mu_j\sinh\mu_j} = \prod_{j=1}^{N} \exp(-q_{1j}\sin^2\theta) \prod_{j=1}^{N-1} (1 - q_{2j}\cos^2\theta) = g_N(\theta, q). \tag{39}$$

Taking advantage of (39) and repeatedly using the formula [5]

$$\int_0^\infty \exp[\mathrm{i}a_0x^2] I_\nu(-\mathrm{i}b_0x) I_\nu(-\mathrm{i}c_0x)x\,\mathrm{d}x = \frac{\mathrm{i}}{2a_0}\exp\left[\frac{-\mathrm{i}}{4a_0}(b_0^2 + c_0^2)\right] I_\nu\left(\frac{-\mathrm{i}b_0c_0}{2a_0}\right) \tag{40}$$

we can use the method developed by [4] to rewrite (36) as

$$\tilde{K}_\lambda = \frac{1}{2\pi\mathrm{i}} \lim_{N\to\infty} a_N g_N(\theta, q)\exp(\mathrm{i}p_N a^2\cosh^2\mu_0 + \mathrm{i}q_N a^2\cosh^2\mu) \times I_\lambda(-\mathrm{i}a_N a^2\cosh\mu_0\cosh\mu) \tag{41}$$

where

$$a_N = \frac{\beta}{N} \qquad p_N = q_N = \frac{\beta}{2N}. \tag{42}$$

Using the formula

$$\int_{-\infty}^{\infty} \delta(u-\phi) f(\phi)\,\mathrm{d}\phi = f(u) \tag{43}$$

we can rewrite (35) as

$$K(\mu,\theta,T;\mu_0,\theta_0,0) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\phi\, \delta(\theta-\theta_0-\phi+2\pi n) \int_{-\infty}^{\infty} \exp[\mathrm{i}(\lambda-2\alpha)\phi]$$
$$\times \frac{1}{2\pi\mathrm{i}} \lim_{N\to\infty} a_N g_N(\theta,q) \exp(\mathrm{i}p_N a^2 \cosh^2\mu_0 + \mathrm{i}q_N a^2\cosh^2\mu)$$
$$\times I_\lambda(-\mathrm{i}a_N a^2 \cosh\mu_0 \cosh\mu)\,\mathrm{d}\lambda. \tag{44}$$

Utilizing (30) we obtain from (44)

$$K = \int_{-\infty}^{\infty} \mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}m(\theta-\theta_0-\phi)+\mathrm{i}(\lambda-2\alpha)\phi] \frac{1}{2\pi\mathrm{i}} \lim_{N\to\infty} a_N g_N(\theta,q)$$
$$\times \exp(\mathrm{i}p_N a^2\cosh^2\mu_0 + \mathrm{i}q_N a^2\cosh^2\mu) I_\lambda(-\mathrm{i}a_N a^2\cosh\mu_0\cosh\mu). \tag{45}$$

When $\lambda \to \lambda + m + 2$, (45) becomes

$$K = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(\mathrm{i}\lambda\phi)\,\mathrm{d}\phi \int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}m(\theta-\theta_0)] \frac{1}{2\pi\mathrm{i}} \lim_{N\to\infty} a_N g_N(\theta,q)$$
$$\times \exp(\mathrm{i}p_N a^2\cosh^2\mu_0 + \mathrm{i}q_N a^2\cosh^2\mu) I_{\lambda+m+2\alpha}(-\mathrm{i}a_N a^2\cosh\mu_0\cosh\mu). \tag{46}$$

Using the formula

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(\mathrm{i}\lambda\phi)\,\mathrm{d}\phi = \delta(\lambda) \tag{47}$$

applying (43), and completing the integration $\int_{-\infty}^{\infty} \delta(\lambda)\ldots\mathrm{d}\lambda$, from (46) we get

$$K(\mu,\theta,T;\mu_0,\theta_0,0) = \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}m(\theta-\theta_0)] \frac{1}{2\pi\mathrm{i}} \lim_{N\to\infty} a_N g_N(\theta,q)$$
$$\times \exp(\mathrm{i}p_N a^2\cosh^2\mu_0 + \mathrm{i}q_N a^2\cosh^2\mu) I_{m+2\alpha}(-\mathrm{i}a_N a^2\cosh\mu_0\cosh\mu). \tag{48}$$

5. Expression of incident plane wave in elliptic cylindrical coordinates

Let τ be the angle between the y axis and the wavevector $\boldsymbol{k}$ of the incident wave, then we can write the wavefunction of the incident wave as

$$\psi_{\mathrm{inc}} = \exp\{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\alpha g(\theta)\} = \exp\{\mathrm{i}k(x\sin\tau + y\cos\tau) - \mathrm{i}\alpha g(\theta)\}. \tag{49}$$

Expressed in elliptic cylindrical coordinates, (49) becomes

$$\psi_{\mathrm{inc}} = \exp\{\mathrm{i}ka(\cosh\mu\cos\theta\sin\tau + \sinh\mu\sin\theta\cos\tau) - \mathrm{i}\alpha g(\theta)\}. \tag{50}$$

Since for the incident wave the current density should be constant and in the $\boldsymbol{k}$ direction, hence we have

$$\boldsymbol{j} = \frac{\hbar}{2\mathrm{i}m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e}{mc}\boldsymbol{A}\psi^*\psi = w\boldsymbol{k} \tag{51}$$

where

$$\boldsymbol{A} = \frac{\Phi}{\pi h}\boldsymbol{e}_\theta \qquad [h \equiv a(\cosh^2\mu - \cos^2\theta)^{1/2}]. \tag{52}$$

Substituting (49) into (51) we get

$$w = \frac{\hbar k}{m} \qquad g'(\theta) = 2 \to g(\theta) = 2\theta. \tag{53}$$

Using the above result $g(\theta) = 2\theta$, and considering the condition (37), from (50) we obtain the following expression for the incident wave in elliptic cylindrical coordinates:

$$\psi(\mu_0, \theta_0, 0) = \exp\{\mathrm{i}ka \cosh\mu_0 \sin(\theta + \tau) - \mathrm{i}2\alpha\theta_0\}. \tag{54}$$

6. Expression of the wavefunction $\psi(\mu, \theta, T)$ at large distances from two fluxes

The wavefunction $\psi(\mu, \theta, T)$ is obtained from

$$\psi(\mu, \theta, T) = \int_0^\infty \int_{-\pi}^{\pi} K(\mu, \theta, T; \mu_0, \theta_0, o) h_{\mu_0} h_{\theta_0}\, \mathrm{d}\mu_0\, \mathrm{d}\theta_0 \tag{55}$$

where the metric coefficients $h_\mu = h_\theta = h = a(\cosh^2\mu - \cos^2\theta)^{1/2}$. At large distances from two fluxes

$$h_{\mu_0} h_{\theta_0}\, \mathrm{d}\mu_0\, \mathrm{d}\theta_0 = a^2(\cosh^2\mu_0 - \cos^2\theta_0)\, \mathrm{d}\mu_0\, \mathrm{d}\theta_0 \approx a^2 \cosh\mu_0 \sinh\mu_0\, \mathrm{d}\mu_0\, \mathrm{d}\theta_0$$
$$= a\cosh\mu_0\, \mathrm{d}(a\cosh\mu_0)\, \mathrm{d}\theta_0. \tag{56}$$

Substituting (48), (54) and (56) into (55) we get

$$\psi(\mu, \theta, T) = \sum_{m=-\infty}^{\infty} \exp(\mathrm{i}m\theta) \frac{1}{\mathrm{i}} \int_0^\infty a\cosh\mu_0\, \mathrm{d}(a\cosh\mu_0) \lim_{N\to\infty} a_N g_N(\theta, q)$$
$$\times \exp(\mathrm{i}p_N a^2 \cosh^2\mu_0 + \mathrm{i}q_N a^2 \cosh^2\mu) I_{m+2\alpha}(-\mathrm{i}a_N a^2 \cosh\mu_0 \cosh\mu)$$
$$\times \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[\mathrm{i}ka\cosh\mu_0 \sin(\theta_0 + \tau) - \mathrm{i}(m + 2\alpha)\theta_0]\, \mathrm{d}\theta_0. \tag{57}$$

Using (40), (42) and the formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-\mathrm{i}z\cos\theta - \mathrm{i}\nu\theta]\, \mathrm{d}\theta = I_\nu(-\mathrm{i}z) \tag{58}$$

neglecting phase factors which are irrelevant to m, and considering

$$\lim_{N\to\infty} g_N(\theta, q) = g(\theta, q) \tag{59}$$

from (57) we obtain

$$\psi(\mu, \theta, T) = \sum_{m=-\infty}^{\infty} \exp\left[\mathrm{i}m\left(\theta + \tau + \frac{\pi}{2}\right)\right] g(\theta, q) I_{m+2\alpha}(-\mathrm{i}ka\cosh\mu). \tag{60}$$

From the asymptotic expansion formula 9.7.1 of [6] we have

$$I_{m+2\alpha}(-\mathrm{i}ka\cosh\mu) \approx I(\mu) = \frac{\exp(-\mathrm{i}ka\cosh\mu)}{\sqrt{-2\pi\mathrm{i}ka\cosh\mu}} \tag{61}$$

hence we can rewrite (60) as

$$\psi(\mu, \theta, T) = I(\mu) \sum_{m=-\infty}^{\infty} \exp\left[\mathrm{i}m\left(\theta + \tau + \frac{\pi}{2}\right)\right] g(\theta, q). \tag{62}$$

Firstly let us calculate $g(\theta, q)$. Considering

$$\cosh\mu_0 \gg 1 \qquad \Delta\mu \approx 1 \tag{63}$$

we have

$$\cosh\mu_j = \cosh(\mu_0 + j\Delta\mu) \approx \tfrac{1}{2}\mathrm{e}^{\mu_0+j\Delta\mu} \approx \tfrac{1}{2}\mathrm{e}^{\mu_0}(1+j\Delta\mu) \approx \tfrac{1}{2}\mathrm{e}^{\mu_0}\Delta\mu(j+1)$$
$$\times \sinh\mu_j \approx \tfrac{1}{2}\mathrm{e}^{\mu_0}\Delta\mu(j-1). \tag{64}$$

Substituting (64) into (38) we get

$$q_{2j} = \frac{q}{2(j+1)(j-1)} = \frac{q}{4}\left[\frac{1}{j-1} - \frac{1}{j+1}\right] \qquad j \neq \pm 1 \qquad \left(q = \frac{8}{\mathrm{e}^{2\mu_0}(\Delta\mu)^2}\right). \tag{65}$$

When $j = \pm 1$, we notice that $|\cosh\mu_j - \sinh\mu_j|$ is large enough such that

$$\cosh^2\mu_1 - \cosh\mu_1\sinh\mu_1 > \cos^2\theta \rightarrow \frac{\cosh^2\mu_1 - \cos^2\theta}{\cosh\mu_1\sinh\mu_1} > 1 \tag{66}$$

so then from (38) we can see that $q_{2,\pm 1}$ must be negative, and we have

$$q_{2,\pm 1} \approx \frac{-1}{\cosh\mu_1\sinh\mu_1} = \frac{-1}{\cosh(\mu_0+\Delta\mu)\sinh(\mu_0+\Delta\mu)} \approx \frac{-q}{8}. \tag{67}$$

(65) and (67) can be rewritten as

$$q_{2j} = \frac{q}{4}\left[\left(\frac{1}{j-1}\right)_{+1} - \left(\frac{1}{j+1}\right)_{-1}\right] \tag{68}$$

where

$$\left(\frac{1}{j-1}\right)_{+1} = \begin{cases} \dfrac{1}{j-1} & j \neq +1 \\ 0 & j = +1 \end{cases} \qquad \left(\frac{1}{j+1}\right)_{-1} = \begin{cases} \dfrac{1}{j+1} & j \neq -1 \\ 0 & j = -1. \end{cases} \tag{69}$$

Introducing

$$q_1 = \lim_{N\to\infty}\sum_{j=1}^{N} q_{1j} \qquad q_2 = \lim_{N\to\infty}\sum_{j=1}^{N-1} q_{2j} \tag{70}$$

and considering

$$(1 - q_{2j}\cos^2\theta) \approx \exp(-q_{2j}\cos^2\theta) \tag{71}$$

from (59) and (39) we have

$$g(\theta, q) = \exp(-q_1\sin^2\theta - q_2\cos^2\theta) = 1 - q_1\sin^2\theta - q_2\cos^2\theta$$
$$= 1 - \frac{q_1+q_2}{2} + \frac{q_1-q_2}{4}(\mathrm{e}^{2\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta}). \tag{72}$$

Substituting (72) into (62) and using the formula

$$\sum_{m=-\infty}^{\infty}\exp\left[\mathrm{i}m\left(\theta+\tau+\frac{\pi}{2}\right)\right]a_m\mathrm{e}^{\mathrm{i}l\theta} = \sum_{m=-\infty}^{\infty}\exp\left[\mathrm{i}m\left(\theta+\tau+\frac{\pi}{2}\right)\right]a_{m-l} \tag{73}$$

we can rewrite the summation term on the right-hand side of (62) as

$$\sum_{m=-\infty}^{\infty}\exp\left[\mathrm{i}m\left(\theta+\tau+\frac{\pi}{2}\right)\right]g(\theta,q) = \sum_{m=-\infty}^{\infty}\exp\left[\mathrm{i}m\left(\theta+\tau+\frac{\pi}{2}\right)\right]$$
$$\times\left\{1 - \frac{q_1+q_2}{2} + \frac{q_1-q_2}{4}2\right\} = \sum_{m=-\infty}^{\infty}\exp\left[\mathrm{i}m\left(\theta+\tau+\frac{\pi}{2}\right)\right]\{1-q_2\}. \tag{74}$$

Substituting (68) into (70) we get

$$q_2 = \lim_{N\to\infty} \sum_{j=1}^{N-1} \frac{q}{4} \left[\left(\frac{1}{j-1} \right)_{+1} - \left(\frac{1}{j+1} \right)_{-1} \right]. \tag{75}$$

When we substitute (75) into (74) we find that the main contribution comes from the term $j = m$ (see the appendix), hence we can write (75) as

$$q_2 = \frac{q}{4} \left[\left(\frac{1}{m-1} \right)_{+1} - \left(\frac{1}{m+1} \right)_{-1} \right]. \tag{76}$$

Substituting (76) into (74) and using (73) we obtain

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} \exp\left[\mathrm{i}m\left(\theta + \tau + \frac{\pi}{2}\right)\right] g(\theta, q) &= \sum_{m=-\infty}^{\infty} \exp\left[\mathrm{i}m\left(\theta + \tau + \frac{\pi}{2}\right)\right] \\
&\quad \times \left\{ 1 - \frac{q}{4} \left[\left(\frac{1}{m-1} \right)_{+1} - \left(\frac{1}{m+1} \right)_{-1} \right] \right\} \\
&= \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\tau+\pi/2)} \left\{ \mathrm{e}^{\mathrm{i}m\theta} - \frac{q}{4} \frac{\mathrm{e}^{\mathrm{i}(m+2)\theta}}{(m+2)-1} + \frac{q}{4} \frac{\mathrm{e}^{\mathrm{i}(m-2)\theta}}{(m-2)+1} \right\} \\
&= \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\tau+\pi/2)} \left\{ \mathrm{e}^{\mathrm{i}m\theta} - \frac{q}{4} \left[\mathrm{e}^{\mathrm{i}(m+2)\theta} \left(\frac{1}{m+1} \right)_{-1} - \mathrm{e}^{\mathrm{i}(m-2)\theta} \left(\frac{1}{m-1} \right)_{+1} \right] \right\}.
\end{aligned} \tag{77}$$

Using the formulae [7]

$$\begin{aligned}
ce_\nu(\theta, q) &= \cos \nu\theta - \frac{q}{4} \left[\frac{\cos(\nu+2)\theta}{\nu+1} - \frac{\cos(\nu-2)\theta}{\nu-1} \right] + \mathrm{O}(q^2) \\
se_\nu(\theta, q) &= \sin \nu\theta - \frac{q}{4} \left[\frac{\sin(\nu+2)\theta}{\nu+1} - \frac{\sin(\nu-2)\theta}{\nu-1} \right] + \mathrm{O}(q^2)
\end{aligned} \tag{78}$$

and noting that

$$ce_{-\nu}(\theta, q) = ce_\nu(\theta, q) \qquad se_{-\nu}(\theta, q) = -se_\nu(\theta, q) \tag{79}$$

when $\nu = \pm 1$ we have

$$\begin{aligned}
ce_{\pm 1}(\theta, q) &= \cos\theta - q\tfrac{1}{8}\cos 3\theta + \mathrm{O}(q^2) \\
se_{\pm 1}(\theta, q) &= \pm\left(\sin\theta - q\tfrac{1}{8}\sin 3\theta\right) + \mathrm{O}(q^2).
\end{aligned} \tag{80}$$

Neglecting $\mathrm{O}(q^2)$ and substituting (78)–(80) into (77) we get

$$\sum_{m=-\infty}^{\infty} \exp\left[\mathrm{i}m\left(\theta + \tau + \frac{\pi}{2}\right)\right] g(\theta, q) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\tau+\pi/2)} \{ce_m(\theta, q) + \mathrm{i}se_m(\theta, q)\}. \tag{81}$$

Substituting (81) into (62) we obtain

$$\psi(\mu, \theta, T) = \sum_{m=-\infty}^{\infty} \{ce_m(\theta, q) + \mathrm{i}se_m(\theta, q)\} \mathrm{e}^{\mathrm{i}m(\tau+\pi/2)} I(\mu). \tag{82}$$

Substituting (61) into (82) we have

$$\psi(\mu, \theta, T) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\tau+\pi/2)} [I_{m+2\alpha}(-\mathrm{i}ka\cosh\mu) + \mathrm{O}(q)] \{ce_m(\theta, q) + \mathrm{i}se_m(\theta, q)\}. \tag{83}$$

Using (79) and the formula

$$I_\nu(x) = \mathrm{e}^{-\nu\pi \mathrm{i}/2} J_\nu(x\mathrm{e}^{\pi \mathrm{i}/2}) \tag{84}$$

from (83) we get

$$\begin{aligned}\psi(\mu,\theta,T) = &\sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}m\tau}[J_{m+2\alpha}(ka\cosh\mu)+\mathrm{O}(q)]\{ce_m(\theta,q)+\mathrm{i}se_m(\theta,q)\} \\ &+\sum_{m=1}^{\infty}(-1)^m \mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}m\tau}[J_{m-2\alpha}(ka\cosh\mu)+\mathrm{O}(q)]\{ce_m(\theta,q)-\mathrm{i}se_m(\theta,q)\}.\end{aligned} \tag{85}$$

Using the asymptotic relation

$$J_{m\pm2\alpha}(k\rho) = \tfrac{1}{2}\mathrm{e}^{\mp\mathrm{i}\alpha\pi}[J_m(k\rho)+\mathrm{i}Y_m(k\rho)] + \tfrac{1}{2}\mathrm{e}^{\pm\mathrm{i}\alpha\pi}[J_m(k\rho)-\mathrm{i}Y_m(k\rho)] \tag{86}$$

and considering that at large distances from the fluxes

$$\rho \approx a\cosh\mu \qquad \phi\approx\theta \tag{87}$$

we can rewrite (85) as

$$\begin{aligned}\psi = &\{\tfrac{1}{2}(\mathrm{e}^{-\mathrm{i}2\alpha\pi}+1)[J_0(k\rho)+\mathrm{O}(q)] + \tfrac{1}{2}\mathrm{i}(\mathrm{e}^{-\mathrm{i}2\alpha\pi}-1)[Y_0(k\rho)+\mathrm{O}(q)]\}ce_0(\theta,q) \\ &+\sum_{n=1}^{\infty}\{2\cos(2\pi\tau-\alpha\pi)\cos(\alpha\pi)[J_{2n}(k\rho)+\mathrm{O}(q)] \\ &+\mathrm{i}2\sin(2n\tau-\alpha\pi)\sin(\alpha\pi)[Y_{2n}(k\rho)+\mathrm{O}(q)]\}ce_{2n}(\theta,q) \\ &+\sum_{n=0}^{\infty}\{\mathrm{i}2\sin[(2n+1)\tau-\alpha\pi]\cos(\alpha\pi)[J_{2n+1}(k\rho)+\mathrm{O}(q)] \\ &+2\cos[(2n+1)\tau-\alpha\pi]\sin(\alpha\pi)[Y_{2n+1}(k\rho)+\mathrm{O}(q)]\}ce_{2n+1}(\theta,q) \\ &+\sum_{n=0}^{\infty}\{\mathrm{i}2\cos[(2n+1)\tau-\alpha\pi]\cos(\alpha\pi)[J_{2n+1}(k\rho)+\mathrm{O}(q)] \\ &-2\sin[(2n+1)\tau-\alpha\pi]\sin(\alpha\pi)[Y_{2n+1}(k\rho)+\mathrm{O}(q)]se_{2n+1}(\theta,q) \\ &+\sum_{n=0}^{\infty}\{-2\sin[(2n+2)\tau-\alpha\pi]\cos(\alpha\pi)[J_{2n+2}(k\rho)+\mathrm{O}(q)] \\ &+\mathrm{i}2\cos[(2n+2)\tau-\alpha\pi]\sin(\alpha\pi)[Y_{2n+2}(k\rho)+\mathrm{O}(q)]se_{2n+2}(\theta,q).\end{aligned} \tag{88}$$

Since at large distances

$$\begin{aligned}&Ce_l(\mu,q)/p_l' \to J_l(k\rho) \qquad l\geqslant 0 \qquad Se_l(\mu,q)/s_l' \to J_l(k\rho), \qquad l\geqslant 1 \\ &Fey_l(\mu,q)/p_l' \to Y_l(k\rho), \qquad l\geqslant 0 \qquad Gey_l(\mu,q)/s_l' \to Y_l(k\rho), \qquad l\geqslant 1\end{aligned} \tag{89}$$

where Ce_l and Fey_l are non-periodic Mathieu functions corresponding to ce_l, and Se_l and Gey_l are non-periodic Mathieu functions corresponding to se_l, hence the terms $[J_l(k\rho)+\mathrm{O}(q)][Y_l(k\rho)+\mathrm{O}(q)]$ are the asymptotic expressions of the corresponding non-periodic Mathieu functions. Therefore, (88) is in complete agreement with equation (32) of [1], which is obtained by directly solving the Schrödinger equation. Using the same method as that used in [1, 2], from (88) we readily obtain the same expression for the AB scattering cross section as before.

Thus, we derive in a very simple way the wavefunction far from the two magnetic flux lines by the path integral approach. Splitting the wavefunction into incident plane waves and the scattered wave, we obtain the AB scattering cross section. Since the obtained wavefunction is completely the same as that obtained by directly solving the Schrödinger equation, the obtained AB scattering cross section is also completely the same as that given in [1, 2]. Therefore, our calculation undoubtedly shows the correctness of our previous calculation and once more replies to the criticism about this matter.

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Appendix

We write the terms other than $j = m$ in (75) as

$$q_2^0 = \lim_{N\to\infty}\sum_{j\neq m}\frac{q}{4}\left[\left(\frac{1}{j-1}\right)_{+1} - \left(\frac{1}{j+1}\right)_{-1}\right] = \sum_{j=1}^{m-1}\frac{q}{4}\left[\left(\frac{1}{j-1}\right)_{+1} - \left(\frac{1}{j+1}\right)_{-1}\right]$$
$$+ \lim_{N\to\infty}\sum_{j=m+1}^{N-1}\frac{q}{4}\left[\left(\frac{1}{j-1}\right)_{+1} - \left(\frac{1}{j+1}\right)_{-1}\right]$$
$$= \frac{q}{4}\left\{1 - \frac{1}{m-1} + \frac{1}{m+1}\right\}. \qquad \text{(A1)}$$

Now we are going to analyse the contribution of q_2^0 in (74):

$$I = \sum_{m=-\infty}^{\infty}\exp\left[\mathrm{i}m\left(\theta+\tau+\frac{\pi}{2}\right)\right]q_2^0. \qquad \text{(A2)}$$

Using (A1) and

$$\sum_{m=0}^{\infty}\mathrm{e}^{\mathrm{i}m\phi} = \frac{1}{1-\mathrm{e}^{\mathrm{i}\phi}} \qquad \sum_{m=-\infty}^{-1}\mathrm{e}^{\mathrm{i}m\phi} = \frac{\mathrm{e}^{-\mathrm{i}\phi}}{1-\mathrm{e}^{-\mathrm{i}\phi}} \qquad \text{(A3)}$$

letting $\phi = \theta + \tau + \pi/2$, we can compute (A2) as follows:

$$I = \sum_{m=-\infty}^{\infty}\exp(\mathrm{i}m\phi)\frac{q}{4}\left\{1 - \frac{1}{m-1} + \frac{1}{m+1}\right\}$$
$$= \frac{q}{4}\left\{\sum_{m=0}^{\infty}\mathrm{e}^{\mathrm{i}m\phi} + \sum_{m=-\infty}^{-1}\mathrm{e}^{\mathrm{i}m\phi} - \mathrm{e}^{\mathrm{i}\phi}\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}(m-1)\phi}}{m-1} + \mathrm{e}^{-\mathrm{i}\phi}\frac{\mathrm{e}^{\mathrm{i}(m+1)\phi}}{m+1}\right\}$$
$$= \frac{q}{4}\left\{\frac{1}{1-\mathrm{e}^{\mathrm{i}\phi}} + \frac{\mathrm{e}^{-\mathrm{i}\phi}}{1-\mathrm{e}^{-\mathrm{i}\phi}} - \mathrm{e}^{\mathrm{i}\phi}\int\sum_{m=-\infty}^{\infty}\mathrm{i}\mathrm{e}^{\mathrm{i}(m-1)\phi}\,\mathrm{d}\phi\right.$$
$$+\mathrm{e}^{-\mathrm{i}\phi}\int\sum_{m=-\infty}^{\infty}\mathrm{i}\mathrm{e}^{\mathrm{i}(m+1)\phi}\,\mathrm{d}\phi = \frac{q}{4}\left\{0 - \mathrm{e}^{\mathrm{i}\phi}\int\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\left[\frac{1}{1-\mathrm{e}^{\mathrm{i}\phi}} + \frac{\mathrm{e}^{-\mathrm{i}\phi}}{1-\mathrm{e}^{-\mathrm{i}\phi}}\right]\mathrm{d}\phi\right.$$
$$\left.+\mathrm{e}^{-\mathrm{i}\phi}\int\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\left[\frac{1}{1-\mathrm{e}^{\mathrm{i}\phi}} + \frac{\mathrm{e}^{-\mathrm{i}\phi}}{1-\mathrm{e}^{-\mathrm{i}\phi}}\right]\mathrm{d}\phi\right\} = 0. \qquad \text{(A4)}$$

Hence only the term $j = m$ has a contribution to (74).

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